Harmonic maps and para-Sasakian geometry

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Abstract. The purpose of this paper is to study the harmonicity of maps to or from para-Sasakian manifolds. We derive the condition for the tension field of paraholomorphic map between almost para-Hermitian manifold and para-Sasakian manifold. The necessary and sufficient condition for a paraholomorphic map between para-Sasakian manifolds to be parapluriharmonic are shown and a non-trivial example is presented for its illustrations.

1. Introduction

The study of harmonic maps was initiated by F. B. Fuller, J. Nash and J. H. Sampson [7, 22] while the first general result on the existence of harmonic maps is due to Eells-Sampson [8]. Harmonic maps are extrema (critical points) of the energy functional defined on the space of smooth maps between Riemannian (pseudo-Riemannian) manifolds. The trace of the second fundamental form of such maps vanishes.

More precisely, let (M_i, g_i), $i \in \{1, 2\}$ be pseudo-Riemannian manifolds and $\Gamma(TM_i)$ denotes the sections of the tangent bundle TM_i of M_i, that is, the space of vector fields on M_i. Then *energy* $E(f)$ of a smooth map $f : (M_1, g_1) \to (M_2, g_2)$ is defined by the formula

$$E(f) = \int_{M_1} e(f) \mathcal{V}_{g_1}, \tag{1.1}$$

where $\mathcal{V}_{g_1}$ is the volume measure associated to the metric g_1 and the *energy density* $e(f)$ of f is the smooth function $e(f) : M_1 \to [0, \infty)$ given by

$$e(f)_p = \frac{1}{2}\|f_*\|^2 = \frac{1}{2}Tr_{g_1}(f^*g_2)(p), \qquad (1.2)$$

for each $p \in M_1$. In the above equation f_* is a linear map $f_* : \Gamma(TM_1) \to \Gamma_f(TM_2)$ therefore it can be considered as a section of the bundle

$$TM_1 \otimes f^{-1}(TM_2) \to M_1,$$

where $f^{-1}(TM_2)$ is the pullback bundle having fibres $(f^{-1}(TM_2))_p = T_{f(p)}M_2$, $p \in M_1$ and f^*g_2 is the pullback metric on M_1. If we denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections on M_1 and M_2 respectively, then the second fundamental form of f is the symmetric map $\alpha_f : \Gamma(TM_1) \times \Gamma(TM_2) \to \Gamma_f(TM_2)$ defined by

$$\alpha_f(X, Y) = \widetilde{\nabla}_X f_* Y - f_* \nabla_X Y, \qquad (1.3)$$

for any $X, Y \in \Gamma(TM_1)$. Where $\widetilde{\nabla}$ is the pullback of the Levi-Civita connection $\overline{\nabla}$ of M_2 to the induced vector bundle $f^{-1}(TM_2) : \widetilde{\nabla}_X f_* Y = \overline{\nabla}_{f_* X} f_* Y$. The section $\tau(f) \in \Gamma(f^{-1}(TM_2))$, defined by

$$\tau(f) = Tr_{g_1}\alpha_f \qquad (1.4)$$

is called the *tension field* of f and a map is said to be harmonic if its tension field vanishes identically (see [4, 9]).

If we consider $\{f_{s,t}\}_{s,t \in (-\epsilon, \epsilon)}$ a smooth two-parameter variation of f such that $f_{0,0} = f$ and let $V, W \in \Gamma(f^{-1}(TM_2))$ be the corresponding variational vector fields then

$$V = \frac{\partial}{\partial s}\left(f_{s,t}\right)\bigg|_{(s,t)=(0,0)}, \quad W = \frac{\partial}{\partial t}\left(f_{s,t}\right)\bigg|_{(s,t)=(0,0)}.$$

The *Hessian* of a harmonic map f is defined by:

$$H_f(V, W) = \frac{\partial^2}{\partial s \partial t}\left(E(f_{s,t})\right)\bigg|_{(s,t)=(0,0)}.$$

The index of a harmonic map $f : (M_1, g_1) \to (M_2, g_2)$ is defined as the dimension of the tangent subspace of $\Gamma(f^{-1}(TM_2))$ on which the Hessian H_f is negative definite. A harmonic map f is said to be *stable* if Morse index (*i.e.*, the dimension of largest subspace of $\Gamma(f^{-1}(TM_2))$ on which the Hessian H_f is negative definite) of f is zero and otherwise, it is said to be *unstable* (see [10, 14]). For a non-degenerate point $p \in M_1$, we decompose the space $T_p M_1$ into its *vertical space* $\nu_p = \ker f_{*p}$ and its *horizontal space* $\mathcal{H}_p = (\ker f_{*p})^\perp$, that is, $\mathcal{H}_p = \nu_p^\perp$, so that $T_p M_1 = \nu_p \oplus \mathcal{H}_p$. The map is said to be horizontally conformal if for each $p \in M_1$ either the rank of f_{*p} is zero (that is, p is a critical point), or the restriction of f_{*p} to the horizontal space $\mathcal{H}_p$ is surjective and conformal (here p is a regular point) [5, 9].

The premise of harmonic maps has acknowledged several important contributions and has been successfully applied in computational fluid dynamics

(CFD), minimal surface theory, string theory, gravity and quantum field theory (see [3, 17, 18, 20]). Most of works on harmonic maps are between Riemannian manifolds [2]. The harmonic maps between pseudo-Riemannian manifolds behave differently and their study must be subject to some restricted classes of pseudo-Riemannian manifolds [6].

This paper is organized as follows. In Sect. 2, the basic definitions about almost para-Hermitian manifolds, almost paracontact manifolds and normal almost paracontact manifolds are given. In Sect. 3, we define and study para-holomorphic map. We prove that the tension field of any (J, ϕ)-paraholomorphic map between almost para-Hermitian manifold and para-Sasakian manifold lies in $\Gamma(D_1)$. Sect. 4 deals with parapluriharmonic map in which we obtain the necessary and sufficient condition for a (ϕ_1, ϕ_2)-paraholomorphic map between para-Sasakian manifolds to be ϕ_1-parapluriharmonic and give an example for its illustrations.

2. Preliminaries

2.1. Almost para-complex manifolds

A smooth manifold N^{2m} of dimension $2m$ is said to be an almost product structure if it admits a tensor field J of type $(1, 1)$ satisfying:

$$J^2 = Id. \tag{2.1}$$

In this case the pair (N^{2m}, J) is called an almost product manifold. An almost para-complex manifold is an almost product manifold (N^{2m}, J) such that the eigenbundles $T^{\pm}N^{2m}$ associated with the eigenvalues ± 1 of tensor field J have the same rank [12]. An almost para-Hermitian manifold $N^{2m}(J, h)$ is a smooth manifold endowed with an almost para-complex structure J and a pseudo-Riemannian metric h compatible in the sense that

$$h(JX, Y) = -h(X, JY), \qquad \forall\, X, Y \in \Gamma(TN^{2m}). \tag{2.2}$$

It follows that the metric h has signature (m, m) and the eigenbundles $T^{\pm}N^{2m}$ are totally isotropic with respect to h. Let $\{e'_1, \cdots, e'_m, e'_{m+1} = Je'_1, \cdots, e'_{2m} = Je'_m\}$ be an orthonormal basis and denote $\epsilon'_i = g(e'_i, e'_i) = \pm 1$: $\epsilon'_i = 1$ for $i = 1, \cdots, m$ and $\epsilon'_i = -1$ for $i = m+1, \cdots, 2m$. The fundamental 2-form of almost para-Hermitian manifold is defined by

$$\Phi(X, Y) = h(JX, Y) \tag{2.3}$$

and the co-differential $\delta\Phi$ of Φ is given as follows

$$(\delta\Phi)(X) = \sum_{i=1}^{2m} \epsilon'_i \left(\nabla_{e'_i}\Phi\right)(e'_i, X). \tag{2.4}$$

An almost para-Hermitian manifold is called para-Kähler if $\nabla J = 0$ [12].

2.2. Almost paracontact metric manifolds

A C^∞ smooth manifold M^{2n+1} of dimension $(2n+1)$ is said to have a triplet (ϕ, ξ, η)-structure if it admits an endomorphism ϕ, a unique vector field ξ and a 1-form η satisfying:

$$\phi^2 = Id - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1, \tag{2.5}$$

where Id is the identity transformation; and the endomorphism ϕ induces an almost paracomplex structure on each fibre of $\ker \eta$, the contact subbundle, *i.e.*, eigen distributions $(\ker \eta)^{\pm 1}$ corresponding to the characteristic values ± 1 of ϕ have equal dimension n.

From the equation (2.5), it can be easily deduced that

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \quad \text{and} \quad \text{rank}(\phi) = 2n. \tag{2.6}$$

This triplet structure (ϕ, ξ, η) is called an almost paracontact structure and the manifold M^{2n+1} equipped with the (ϕ, ξ, η)-structure is called an almost paracontact manifold (see also [15, 19, 21]). If an almost paracontact manifold admits a pseudo-Riemannian metric g satisfying:

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.7}$$

where signature of g is necessarily $(n+1, n)$ for any vector fields X and Y; then the quadruple (ϕ, ξ, η, g) is called an almost paracontact metric structure and the manifold M^{2n+1} equipped with paracontact metric structure is called an almost paracontact metric manifold. With respect to g, η is metrically dual to ξ, that is

$$g(X, \xi) = \eta(X). \tag{2.8}$$

Also, equation (2.7) implies that

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.9}$$

Further, in addition to the above properties, if the structure-(ϕ, ξ, η, g) satisfies:

$$d\eta(X, Y) = g(X, \phi Y),$$

for all vector fields X, Y on M^{2n+1}, then the manifold is called a paracontact metric manifold and the corresponding structure-(ϕ, ξ, η, g) is called a paracontact structure with the associated metric g [24]. For an almost paracontact metric manifold, there always exists a special kind of local pseudo-orthonormal basis $\{X_i, X_{i^*}, \xi\}$; where $X_{i^*} = \phi X_i$; ξ and X_i's are space-like vector fields and X_{i^*}'s are time-like. Such a basis is called a ϕ-basis. Hence, an almost paracontact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is an odd dimensional manifold with a structure group $\mathbb{U}(n, \mathbb{R}) \times Id$, where $\mathbb{U}(n, \mathbb{R})$ is the para-unitary group isomorphic to $\mathbb{GL}(n, \mathbb{R})$.

An almost paracontact metric structure-(ϕ, ξ, η, g) is para-Sasakian if and only if

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X. \tag{2.10}$$

From Eqs. (2.6), (2.9) and (2.10), it can be easily deduced for a para-Sasakian manifold that

$$\nabla_X \xi = -\phi X, \quad \nabla_\xi \xi = 0. \tag{2.11}$$

In particular, a para-Sasakian manifold is K-paracontact [24].

2.3. Normal almost paracontact metric manifolds

On an almost paracontact metric manifold, one defines the $(1,2)$-tensor field N_ϕ by

$$N_\phi := [\phi, \phi] - 2\, d\eta \otimes \xi, \tag{2.12}$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. If N_ϕ vanishes identically, then we say that the manifold M^{2n+1} is a normal almost paracontact metric manifold [16, 24]. The normality condition implies that the almost paracomplex structure J defined on $M^{2n+1} \times \mathbb{R}$ by

$$J\left(X, \lambda \frac{d}{dt}\right) = \left(\phi X + \lambda \xi, \eta(X) \frac{d}{dt}\right)$$

is integrable. Here X is tangent to M^{2n+1}, t is the coordinate on $\mathbb{R}$ and λ is a C^∞ function on $M^{2n+1} \times \mathbb{R}$. Now we recall the following proposition which characterized the normality of almost paracontact metric 3-manifolds:

Proposition 2.1. [23] *For an almost paracontact metric 3-manifold M^3, the following three conditions are mutually equivalent*

(i) *M^3 is normal,*

(ii) *there exist smooth functions p, q on M^3 such that*

$$(\nabla_X \phi)Y = q(g(X,Y)\xi - \eta(Y)X) + p(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.13}$$

(iii) *there exist smooth functions p, q on M^3 such that*

$$\nabla_X \xi = p(X - \eta(X)\xi) + q\phi X, \tag{2.14}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric g.

The functions p, q appearing in Eqs. (2.13) and (2.14) are given by

$$2p = trace\left\{X \to \nabla_X \xi\right\}, \quad 2q = trace\left\{X \to \phi \nabla_X \xi\right\}. \tag{2.15}$$

A normal almost paracontact metric 3-manifold is called paracosymplectic if $p = q = 0$ and para-Sasakian if $p = 0, q = -1$ [21].

3. Paraholomorphic map

One can look structure preserving mapping between almost para-Hermitian and almost paracontact manifolds as analogous of the well-known holomorphic mappings in complex geometry [1, 11].

Definition 3.1. Let $M_i^{2n_i+1}(\phi_i, \xi_i, \eta_i, g_i)$, $i \in \{1, 2\}$ be almost paracontact metric manifolds and $N^{2m}(J, h)$ be an almost para-Hermitian manifold. Then a smooth map

1. $f : M_1^{2n_1+1} \to N^{2m}$ is (ϕ_1, J)-paraholomorphic map if $f_* \circ \phi_1 = J \circ f_*$. For such a map $f_*\xi_1 = 0$.
2. $f : N^{2m} \to M_1^{2n_1+1}$ is (J, ϕ_1)-paraholomorphic map if $f_* \circ J = \phi_1 \circ f_*$. Here $\mathrm{Im} f_* \perp \xi_1$.
3. $f : M_1^{2n_1+1} \to M_2^{2n_2+1}$ is (ϕ_1, ϕ_2)-paraholomorphic map if $f_* \circ \phi_1 = \phi_2 \circ f_*$. In particular, $f_*(\xi_1^\perp) \subset \xi_2^\perp$ and $f_*(\xi_1) \sim \xi_2$.

When f_* interwines the structures upto a minus sign, we say about (ϕ_1, J)-anti paraholomorphic, (J, ϕ_1)-anti paraholomorphic and (ϕ_1, ϕ_2)-anti paraholomorphic mappings.

Now, we prove the following result.

Proposition 3.2. *Let f be a smooth (ϕ_1, ϕ_2)-paraholomorphic map between para-Sasakian manifolds $M_i^{2n_i+1}(\phi_i, \xi_i, \eta_i, g_i)$, $i \in \{1, 2\}$. Then*

$$\phi_2(\tau(f)) = f_*(\mathrm{div}\phi_1) - Tr_{g_1}\beta, \tag{3.1}$$

where $\beta(X, Y) = (\widetilde{\nabla}_X \phi_2)(f_*Y)$, $\forall\ X, Y \in \Gamma(TM_1^{2n_1+1})$.

Proof. Since f_* has values in $f^{-1}(TM_2^{2n_2+1})$ so that $f_* \circ \phi_1$ and $\phi_2 \circ f_*$ have values in $f^{-1}(TM_2^{2n_2+1})$. Thus, we have

$$\begin{aligned}
(\widetilde{\nabla}(f_* \circ \phi_1))(X, Y) &= \widetilde{\nabla}_X f_*(\phi_1 Y) - (f_* \circ \phi_1)(\nabla_X Y) \\
&= (\widetilde{\nabla}_X f_*)(\phi_1 Y) + f_*(\nabla_X \phi_1 Y) - (f_* \circ \phi_1)(\nabla_X Y) \\
&= \alpha_f(X, \phi_1 Y) + f_*((\nabla\phi_1)(X, Y)). \tag{3.2}
\end{aligned}$$

In the last equality, we have used (1.3). On the other hand, we obtain

$$\begin{aligned}
(\widetilde{\nabla}(\phi_2 \circ f_*))(X, Y) &= \widetilde{\nabla}_X \phi_2(f_*Y) - (\phi_2 \circ f_*)(\nabla_X Y) \\
&= (\widetilde{\nabla}_X \phi_2)(f_*Y) + \phi_2(\widetilde{\nabla}_X f_*Y) - \phi_2(f_*(\nabla_X Y)) \\
&= (\widetilde{\nabla}_X \phi_2)(f_*Y) + \phi_2(\alpha_f(X, Y)). \tag{3.3}
\end{aligned}$$

From Eqs. (3.2) and (3.3), we have

$$\phi_2(\alpha_f(X, Y)) + (\widetilde{\nabla}_X \phi_2)(f_*Y) = f_*((\nabla\phi_1)(X, Y)) + \alpha_f(X, \phi_1 Y). \tag{3.4}$$

Let $\{e_1, e_2, \cdots, e_{n_1}, \phi_1 e_1, \phi_1 e_2, \cdots, \phi_1 e_{n_1}, \xi_1\}$ be a local orthonormal frame for $TM_1^{2n_1+1}$. Taking the trace in (3.4) and using the fact that α_f is symmetric, we have (3.1). This completes the proof. □

Following the proof of the above proposition, we can give the following remarks:

Remark 3.3. For a para-Sasakian manifold $M_1^{2n_1+1}(\phi_1, \xi_1, \eta_1, g_1)$ and a para-Hermitian manifold $N^{2m}(J, h)$. If

(a) $f : M_1^{2n_1+1} \to N^{2m}$ be a (ϕ_1, J)-paraholomorphic map then we have

$$J(\tau(f)) = f_* \mathrm{div}\phi_1 - Tr_{g_1}\beta', \tag{3.5}$$

where $\beta'(X, Y) = (\widetilde{\nabla}_X J)(f_* Y), \qquad \forall\ X, Y \in \Gamma(TM_1^{2n_1+1})$.

(b) $f : N^{2m} \to M_1^{2n_1+1}$ be a (J, ϕ_1)-paraholomorphic map then we have

$$\phi_1(\tau(f)) = f_* \mathrm{div} J - Tr_h \beta'', \tag{3.6}$$

where $\beta''(X, Y) = (\widetilde{\nabla}_X \phi_1)(f_* Y), \qquad \forall\ X, Y \in \Gamma(TN^{2m})$.

Theorem 3.4. *Let f be a (ϕ_1, J)-paraholomorphic map between para-Sasakian manifold $M_1^{2n_1+1}(\phi_1, \xi_1, \eta_1, g_1)$ and para-Kähler manifold $N^{2m}(J, h)$. Then f is harmonic.*

Proof. Let $\{e_1, \cdots, e_{n_1}, \phi_1 e_1, \cdots, \phi_1 e_{n_1}, \xi_1\}$ be a local orthonormal adapted basis on $TM_1^{2n_1+1}$, then from Eqs. (2.6) and (2.10), we have $\mathrm{div}\phi_1 = 0$ (since for a (ϕ_1, J)-paraholomorphic map $f_* \xi_1 = 0$). It follows by the use of equation (3.5) that $J(\tau(f)) = 0$ as N^{2m} is a para-Kähler manifold. Therefore, $\tau(f) = 0$ and f is harmonic. This completes the proof of the theorem. $\square$

For $i \in \{1, 2\}$, let D_i be real distributions, respectively, on para-Sasakian manifolds $M_i^{2n_i+1}$ of rank $2n_i$ then it admits globally defined 1-form η_i such that $D_i \subseteq \ker \eta_i$. Clearly, $TM_i^{2n_i+1} = D_i \oplus \{\xi_i\}$, where $\{\xi_i\}$ is the real distribution of rank one defined by ξ_i [11].
Now, we prove:

Theorem 3.5. *For any (J, ϕ_1)-paraholomorphic map f between almost para-Hermitian manifold $N^{2m}(J, h)$ and para-Sasakian manifold $M_1^{2n_1+1}(\phi_1, \xi_1, \eta_1, g_1)$, the tension field $\tau(f) \in \Gamma(D_1)$.*

Before going to proof of this theorem, we first prove the following proposition:

Proposition 3.6. *For an almost para-Hermitian manifold $N^{2m}(J, h)$, we have*

$$\sum_{i=1}^{m} \left\{ \nabla_{Je_i'} Je_i' - \nabla_{e_i'} e_i' \right\} = J \left\{ \mathrm{div} J - \sum_{i=1}^{m} \left[e_i', Je_i' \right] \right\} \tag{3.7}$$

where $\{e_1', e_2', \cdots, e_m', Je_1', Je_2', \cdots, Je_m'\}$ is a local orthonormal frame on TN^{2m}.

Proof. It is straightforward to calculate

$$\mathrm{div} J = \sum_{i=1}^{m} \left\{ [e_i', Je_i'] - J(\nabla_{e_i'} e_i') + J(\nabla_{Je_i'} Je_i') \right\} \tag{3.8}$$

and the result follows from (2.1) and (3.8). This completes the proof. $\square$

Proof of Theorem 3.5. Since $f_*(X) \in \Gamma(D_1)$, $\forall X \in \Gamma\big(TN^{2m}\big)$ therefore for any local orthonormal frame $\{e_1', e_2', \cdots, e_m', Je_1', Je_2', \cdots, Je_m'\}$ on TN^{2m}, we obtain by using Eqs. (1.3), (1.4), (2.9) and (2.11) that

$$g_1(\tau(f), \xi_1) = \sum_{i=1}^{m} \Big\{ g_1(f_*(\nabla_{Je_i'} Je_i'), \xi_1) - g_1(f_*(\nabla_{e_i'} e_i'), \xi_1) \Big\}. \qquad (3.9)$$

Employing Eq. (3.7), the above equation reduces to

$$g_1(\tau(f), \xi_1) = g_1\left(\phi_1 f_* \left(\mathrm{div} J - \sum_{i=1}^{m} J[e_i', Je_i'] \right), \xi_1 \right). \qquad (3.10)$$

Reusing Eq. (2.9) in (3.10), we get

$$g_1(\tau(f), \xi_1) = 0,$$

which shows that $\tau(f) \in \Gamma(D_1)$. This completes the proof of the theorem. $\square$
By the consequence of the above theorem we can state the following result as a corollary of the theorem 3.5.

Corollary 3.7. *Let* $N^{2m}(J, h)$ *and* $M_1^{2n_1+1}(\phi_1, \xi_1, \eta_1, g_1)$ *be para-Kähler and para-Sasakian manifolds respectively. Then for any* (J, ϕ_1)*-paraholomorphic map* $f :$ $N^{2m} \to M_1^{2n_1+1}$*, the tension field* $\tau(f) \in \Gamma(D_1)$.

4. Parapluriharmonic map

In this section we define the notion of ϕ_1-parapluriharmonic map which is similar to the notion of ϕ-pluriharmonic map between almost contact metric manifold and Riemannian manifold, for ϕ-pluriharmonic map see : [1, 13].

Definition 4.1. A smooth map f between almost paracontact metric manifold $M_1^{2n_1+1}(\phi_1, \xi_1, \eta_1, g_1)$ and pseudo-Riemannian manifold N^m, is said to be ϕ_1-parapluriharmonic if

$$\alpha_f(X, Y) - \alpha_f(\phi_1 X, \phi_1 Y) = 0, \qquad \forall\, X, Y \in \Gamma\big(TM_1^{2n_1+1}\big), \qquad (4.1)$$

where the second fundamental form α_f of f is defined by (1.3). In particular, $\alpha_f(X, \xi_1) = 0$ for any tangent vector X.

Proposition 4.2. *Any* ϕ_1*-parapluriharmonic map* f *between almost paracontact metric manifold* $M_1^{2n_1+1}(\phi_1, \xi_1, \eta_1, g_1)$ *and pseudo-Riemannian manifold* N^m *is harmonic.*

Proof. Let $\{e_1, \cdots, e_{n_1}, \phi_1 e_1, \cdots, \phi_1 e_{n_1}, \xi_1\}$ be a local orthonormal frame on $TM_1^{2n_1+1}$, then by definition of ϕ_1-parapluriharmonicity, we have

$$\alpha_f(\xi_1, \xi_1) = 0 \quad \text{and} \quad \alpha_f(e_i, e_i) - \alpha_f(\phi_1 e_i, \phi_1 e_i) = 0,$$

for $i \in \{1, 2, \cdots, n\}$. Therefore, $\tau(f) = Tr_{g_1}\alpha_f = 0$. This completes the proof.
$\square$

Theorem 4.3. *Let f be a smooth (ϕ_1, J)-paraholomorphic map between normal almost paracontact metric 3-manifold $M_1^3(\phi_1, \xi_1, \eta_1, g_1)$ and para-Kähler manifold $N^2(J, h)$. Then f is harmonic.*

Proof. We recall that $f_*\xi_1 = 0$ for a (ϕ_1, J)-paraholomorphic map and N^2 is para-Kähler, and that from Eq. (2.13) for any vectors X, Y tangent to M_1^3, we have

$$f_*(\nabla_X \phi_1)Y = -\{qf_*X + pf_*\phi_1 X\}\eta_1(Y). \tag{4.2}$$

Using equation (3.4) for a given map, we obtain

$$J(\alpha_f(X, Y)) = -\{qf_*X + pf_*\phi_1 X\}\eta_1(Y) + \alpha_f(X, \phi_1 Y). \tag{4.3}$$

Replacing Y by $\phi_1 Y$ and employing Eqs. (2.5) and (2.6), the above equation reduces to

$$J(\alpha_f(X, \phi_1 Y)) = \alpha_f(X, Y).$$

By the virtue of the fact that α_f is symmetric, we obtain from above equation that

$$\alpha_f(X, Y) - \alpha_f(\phi_1 X, \phi_1 Y) = 0.$$

The above expresion implies that f is ϕ_1-parapluriharmonic and thus harmonic from the proposition 4.2. This completes the proof of the theorem. $\square$

As an immediate consequence of above theorem and remark 2.4 of [21] one easily gets the following corollary:

Corollary 4.4. *Let $M_1^3(\phi_1, \xi_1, \eta_1, g_1)$ be a normal almost paracontact metric 3-manifold with $p, q =$constant, $N^2(J, h)$ be a para-Kähler manifold and $f : M_1^3 \to N^2$ be a smooth (ϕ_1, J)-paraholomorphic map. Then M_1^3 is paracosymplectic manifold.*

Here, we derive the necessary and sufficient condition for a (ϕ_1, ϕ_2)-paraholomorphic map between para-Sasakian manifolds to be ϕ_1-parapluriharmonic.

Theorem 4.5. *Let $f : M_1^{2n_1+1} \to M_2^{2n_2+1}$ be a (ϕ_1, ϕ_2)-paraholomorphic map between para-Sasakian manifolds $M_i^{2n_i+1}(\phi_i, \xi_i, \eta_i, g_i)$, $i \in \{1, 2\}$. Then f is ϕ_1-parapluriharmonic if and only if $\xi_2 \in (Im\, f_*)^{\perp}$.*

Proof. Since f is a (ϕ_1, ϕ_2)-paraholomorphic map then for all $x \in M_1^{2n_1+1}$ there exists a function λ on $M_1^{2n_1+1}$ such that

$$(f_*\xi_1)_{f(x)} = \lambda(x)(\xi_2)_{f(x)} \quad \text{and} \quad (f^*\eta_2)_x = \lambda(x)(\eta_1)_x. \tag{4.4}$$

For any $X, Y \in \Gamma(D_1)$, we have from Eqs. (1.3), (2.10) and (4.4) that

$$\alpha_f(X, \phi_1 Y) = \phi_2 \alpha_f(X, Y) + \eta_2(f_*X)f_*Y - g_2(f_*X, f_*Y)\xi_2 + \lambda g_1(X, Y)\xi_2.$$

From above equation and the fact that α_f is symmetric, we obtain that

$$\alpha_f(X, \phi_1 Y) - \alpha_f(\phi_1 X, Y) = \eta_2(f_*Y)f_*X - \eta_2(f_*X)f_*Y. \tag{4.5}$$

Replacing Y by $\phi_1 Y$ in above expression and using Eqs. (2.5) and (2.6), we find

$$\alpha_f(X,Y) - \alpha_f(\phi_1 X, \phi_1 Y) = -\eta_2(f_* X)\phi_2(f_* Y). \qquad (4.6)$$

This implies that $\alpha_f(X,Y) - \alpha_f(\phi_1 X, \phi_1 Y) = 0$ if and only if $\xi_2 \in (Im\, f_*)^{\perp}$. This completes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, we present an example for illustrating theorem 4.5:

4.1. Example

Let $M_i^3 \subset \mathbb{R}^3$, $i \in \{1,2\}$ be 3-dimensional manifolds with standard Cartesian coordinates. Define the almost paracontact structures $(\phi_i, \xi_i, \eta_i, g_i)$ respectively on M_i^3 by

$$\phi_1 e_1 = -e_2 + x^2 e_3, \ \phi_1 e_2 = -e_1, \ \phi_1 e_3 = 0, \ \xi_1 = e_3, \ \eta_1 = x^2 dy + dz, \qquad (4.7)$$

$$\phi_2 e_1' = -e_2', \ \phi_2 e_2' = -e_1' + v^2 e_3', \ \phi_2 e_3' = 0, \ \xi_2 = e_3', \ \eta_2 = -v^2 du + dw, \quad (4.8)$$

where $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$, $e_1' = \frac{\partial}{\partial u}$, $e_2' = \frac{\partial}{\partial v}$ and $e_3' = \frac{\partial}{\partial w}$. By direct calculations, one verifies that the Nijenhuis torsion of ϕ_i for $i \in \{1,2\}$ vanishes, which implies that the structures are normal. Let the pseudo-Riemannian metrics g_i, $i \in \{1,2\}$ are prescribed respectively on M_i^3 by

$$[g_1(e_s, e_t)] = \begin{bmatrix} -x & 0 & 0 \\ 0 & x^4 + x & x^2 \\ 0 & x^2 & 1 \end{bmatrix}, \ [g_2(e_s', e_t')] = \begin{bmatrix} v^4 + v & 0 & v^2 \\ 0 & -v & 0 \\ v^2 & 0 & 1 \end{bmatrix}, \quad (4.9)$$

for all $s,t \in \{1,2,3\}$. For the Levi-Civita connections $\nabla, \overline{\nabla}$ with respect to metrics g_1, g_2 respectively, we obtain

$$\nabla_{e_1} e_1 = \frac{1}{2x}e_1, \ \nabla_{e_1} e_2 = \frac{2x^3+1}{2x}e_2 + \left(\frac{x}{2} - x^4\right)e_3 = \nabla_{e_2}e_1, \ \nabla_{e_2}e_2 = \frac{4x^3+1}{2x}e_1,$$

$$\nabla_{e_1}e_3 = e_2 - x^2 e_3 = \nabla_{e_3}e_1, \ \nabla_{e_2}e_3 = e_1 = \nabla_{e_3}e_2, \ \nabla_{e_3}e_3 = 0,$$

$$\overline{\nabla}_{e_1'}e_1' = \frac{4v^3+1}{2v}e_2', \ \overline{\nabla}_{e_1'}e_2' = \frac{2v^3+1}{2v}e_1' + \left(\frac{v}{2} - v^4\right)e_3' = \overline{\nabla}_{e_2'}e_1', \ \overline{\nabla}_{e_3'}e_3' = 0,$$

$$\overline{\nabla}_{e_2'}e_3' = e_1' - v^2 e_3' = \overline{\nabla}_{e_3'}e_2', \ \overline{\nabla}_{e_2'}e_2' = \frac{1}{2v}e_2', \ \overline{\nabla}_{e_3'}e_1' = e_2' = \overline{\nabla}_{e_1'}e_3'.$$

From above expressions and equation (2.14), we find $p = 0$, $q = -1$. Hence the M_1^3 and M_2^3 are para-Saakian manifolds with invariant distributions $D_1 = \mathrm{span}\{e_1, \phi_1 e_1\}$ and $D_2 = \mathrm{span}\{e_2', \phi_2 e_2'\}$ respectively. Let $f : M_1^3 \to M_2^3$ be a mapping defined by $f(x,y,z) = (y,x,z)$. Then $f_* \circ \phi_1 = \phi_2 \circ f_*$, i.e., f is a (ϕ_1, ϕ_2)-paraholomorphic map between para-Sasakian manifolds. For any $X, Y \in \Gamma(D_1)$ and $x \in M_1^3$, it is not hard to see that $\alpha_f(X,Y) = \alpha_f(\phi_1 X, \phi_1 Y)$, $\lambda(x) = 1$ and $g_2(\xi_2, f_* X) = 0$. Thus theorem 4.5 is verified.

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